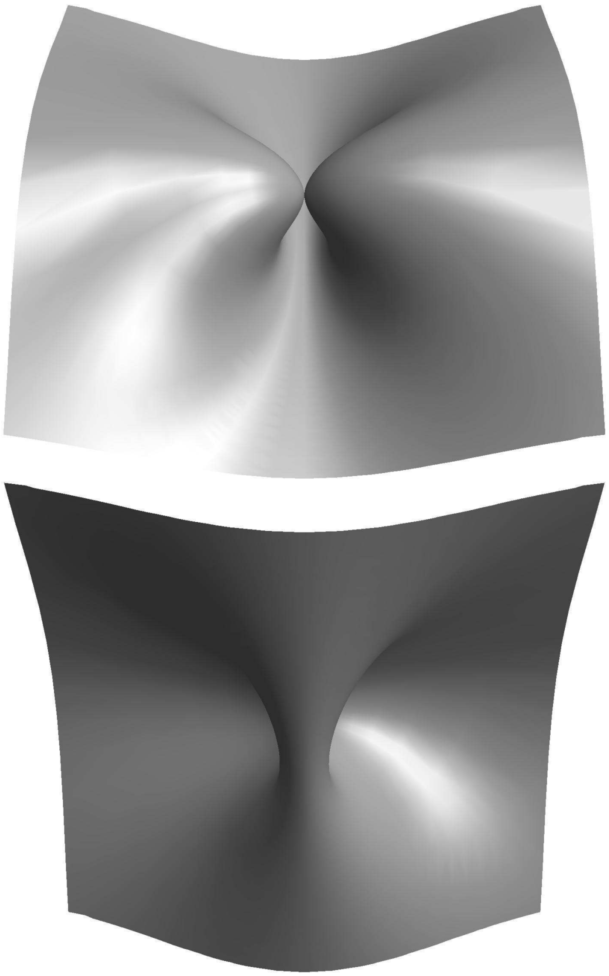

Fluid Membranes I:

Geometry
Variational Principles
Stress and Conservation Laws

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Free Fluid Membranes

- Striking feature of cell membranes:
  behave as fluid membranes on mesoscopic length scales

- Mechanical properties in equilibrium:
  Largely captured by geometrical degrees of freedom of surface
  Fluid $\Rightarrow$ Shear unpenalized

  Described with extraordinary accuracy by Helfrich bending energy
  – or its extensions

- Quadratic in curvature $\Rightarrow$ non-linear field theory:

$$\nabla^2 \text{Curvature} + \text{Curvature}^3 = \text{Sources}$$

- nevermind physics, just getting geometry right is no mean feat

All non-trivial geometries are stressed

- Shape in most interesting bio-membranes controlled by external forces – both global and local

  They themselves play a role in mediating interactions

- Forces are transmitted by the stress established in the membrane

- These stresses are completely determined by the geometry – unusual in a soft matter system

  Thus by examining the geometry it becomes possible to determine the forces that are the source of stress

- In this lecture I will show how to identify this stress tensor and examine a few of its properties

A little geometry

- Parametrized closed surface: $(u^1, u^2) \to \mathbf{X}(u^1, u^2)$

  $\mathbf{X} = (X^1, X^2, X^3)$ 3 functions of 2 variables

- (1) Tangents $\mathbf{e}_a = \partial_a \mathbf{X}$, $a = 1, 2$
  (2) Unit normal $\mathbf{n}$

- Two surface tensors encode geometry:

  (1) Induced metric: $g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b$
  (2) Extrinsic curvature: how fast $\mathbf{n}$ rotates along parameter curves

  Weingarten equation: $\partial_a \mathbf{n} = K_a{}^b \, \mathbf{e}_b$
  $K_{ab} = \mathbf{e}_a \cdot \partial_b \mathbf{n} = g_{ac} K_b{}^c$, $K_{ab} = K_{ba}$

- Monge: $\mathbf{r} \to (\mathbf{r}, h(\mathbf{r}))$
  $K_{ab} = -\partial_a \partial_b h // (1 + |\nabla h|^2)^{3/2} \approx -\partial_a \partial_b h$ w.r.t tangent plane

Gauss equations

- Gauss : a statement about tangent vector change

$$\partial_a \mathbf{e}_b = \Gamma_{ab}{}^c \mathbf{e}_c - K_{ab}\mathbf{n}\,.$$

  Same K_{ab}!; $\Gamma_{ab}{}^c = \Gamma_{ba}{}^c$

- Identify $\Gamma_{ab}{}^c$ as Christoffel symbols associated with the surface covariant derivative ∇_a:

$$\begin{aligned}\Gamma_{ab}{}^c &= g^{cd}\mathbf{e}_d \cdot \partial_a \mathbf{e}_b \\ &= \frac{1}{2} g^{cd} \left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\right) \,.\end{aligned}$$

  Follows directly from the definition $g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b$.

- Introduce covariant derivative $\nabla_a \mathbf{e}_b = \partial_a \mathbf{e}_b - \Gamma_{ab}{}^c \mathbf{e}_c$
  Write Gauss as $\nabla_a \mathbf{e}_b = -K_{ab}\,\mathbf{n}$.

  Why bother: $\nabla_a \mathbf{e}_b$, unlike $\partial_a \mathbf{e}_b$ transforms as a second rank covariant tensor under reparametrization.

Helfrich Hamiltonian

- Bending energy is quadratic in curvature

  Eigenvalues: $K^a{}_b V_I^b = C_I V_I^a \;\; I = 1,2$
  Two symmetric curvature scalars:

  (1) $K := g^{ab} K_{ab} = C_1 + C_2$
  (2) $K_G = \det K^a{}_b = \frac{1}{2}(K^2 - K^{ab}K_{ab}) = C_1 C_2$
  (3) Third degree of freedom is a direction

- Helfrich bending energy (ignore spontaneous curvature):

$$H_0[\mathbf{X}] = \frac{1}{2}\kappa \int dA \; (C_1 + C_2)^2$$

- Topological Gaussian energy: $H_{GB} = \bar{\kappa} \int dA \; C_1 C_2$
  will set boundary conditions!

- Third quadratic: $(C_1 - C_2)^2 = (C_1 + C_2)^2 - 4C_1C_2$

$$H = \frac{1}{2}\int dA\ (C_1 - C_2)^2$$

From bulk point of view H_0 unique!

- Area, enclosed volume, etc fixed using global Lagrange multipliers

Variational Principles I

- $H[\mathbf{X}]$ a functional of $\mathbf{X}$

  $A = \int d^2u \, |\mathbf{e}_1 \times \mathbf{e}_2| = \int d^2u \, \sqrt{g}$
  Lagrange identity: $|\mathbf{e}_1 \times \mathbf{e}_2|^2 = \det g_{ab} = g$

  More generally $H = \int dA \, \mathcal{H}[g_{ab}, K_{ab}]$

- Response to a small deformation of the surface?
  $\mathbf{X}(u^1, u^2) \to \mathbf{X}(u^1, u^2) + \delta\mathbf{X}(u^1, u^2)$

  Decompose: $\delta\mathbf{X} = \Phi^a \, \mathbf{e}_a + \Phi \, \mathbf{n}$

  Tangential part $\delta_{||}\mathbf{X} = \Phi^a \, \mathbf{e}_a = \Phi^a \partial_a \mathbf{X}$
  – Behavior of scalar under change of parameter: $u^a \to u^a - \Phi^a(u^1, u^2)$
  – g_{ab} and K_{ab} respond accordingly

In absense of additional material degrees of freedom: only observable effect is boundary displacement

- Tangential deformation of H

$$\delta_{||} \int dA\, \mathcal{H} = \int dA\, \nabla_a (\mathcal{H}\, \Phi^a)$$

  Need only $\delta_{||}\mathcal{H} = \Phi^a \partial_a \mathcal{H}$
  $\delta_{||} g_{ab} = \nabla_a \Phi_b + \nabla_b \Phi_a$ so that $\delta_{||} \sqrt{g} = \sqrt{g}\, \nabla_a \Phi^a$

- Divergence theorem: $\int dA\, \nabla_a V^a = \int ds\, l_a V^a$
  l_a is unit normal to the boundary out of the surface: $l_a V^a = \mathbf{l} \cdot \mathbf{V}$

  $\delta_{||} H = 0$ on closed surface; no role in bulk defos, equilibrium or not!

- $\delta^2_{||} H$ not so simple

Bianchi-type Identities

- $H = \int dA\, \mathcal{H}[g_{ab}, K_{ab}]$:

  $T^{ab} = -2\delta H/\delta g_{ab}/\sqrt{g}$
  $H^{ab} = \partial\mathcal{H}/\partial K_{ab}$

- Invariance under reparam: $\Leftrightarrow \nabla_a f^{ab} + K^{ab} f_a = 0$

  $f^{ab} = T^{ab} - H^{ac} K_c{}^b$
  $f^a = -\nabla_b H^{ab}$

  Analogue of Bianchi identity for gravity: T^{ab} is not the full story!
  Indepenent of equilibrium!

But not contentless in general ...

- Add material fields: say vector field v^a, $\mathcal{H}[g_{ab}, K_{ab}, v^a]$

  Euler Lagrange $\mathcal{V}_a = \delta H/\delta v^a/\sqrt{g}$

  However, in equilibrium, $\delta_\Phi v^a$ does not contribute:
  $\nabla_a f^{ab} + K^{ab} f_a = 0$ represents conservation of v^a
  whether or not surface is in equilibrium

  If H is independent on K_{ab}: $\nabla_a T^{ab} = 0$.

  Remains to identify appropriate conservation law for $\mathbf{X}$?
  First look at normal deformations

Normal deformations

- Saw that tangent defo does not contribute to Euler-Lagrange derivative
  Normal deformation of $H[\mathbf{X}]$ can be cast as

$$\delta_\perp H[X] = \int dA \ (\mathcal{E}\ \Phi + \nabla_a V^a_\perp) \ .$$

  $\mathcal{E}$ is the (normal) Euler-Lagrange derivative; in equilibrium $\mathcal{E} = 0$

- To identify $\mathcal{E}$, need to peel off derivatives of Φ in the bulk using integration by parts. These derivatives are then collected in a divergence
  – $V^a_\perp$ associated vector field

- Induced normal deformations of g_{ab} and K_{ab}:

  (1) $\delta_\perp g_{ab} = 2 K_{ab} \Phi$
  (2) $\delta_\perp K_{ab} = -(\nabla_a \nabla_b - K_{ac} K^c{}_b)\Phi$

Bending energy and shape equation

- $H[\mathbf{X}] = \frac{1}{2}\int dA\, K^2$

- Eqs.(1) & (2) on previous slide $\Rightarrow$
  $\delta_\perp \sqrt{g} = \sqrt{g} g^{ab} \delta_\perp g_{ab} = \sqrt{g}\, K \Phi$; $\delta_\perp K = -\Delta \Phi - K_{ab} K^{ab} \Phi$

$$\delta_\perp H = \int dA\, K\, [-\nabla^2 \Phi - (K_{ab} - \frac{1}{2} g_{ab} K) K^{ab} \Phi]$$

  Integrate by parts twice $\Rightarrow$ Euler-Lagrange:

$$\mathcal{E} = -\nabla^2 K - K(K_{ab} - \frac{1}{2} g_{ab} K) K^{ab} \,.$$

  $\mathcal{E} = 0$ when $K = 0$ or when $K_{ab} = \frac{1}{2} g_{ab} K$, a sphere
  Area, enclosed volume constant: $\mathcal{E} + \sigma K - P = 0$

- Limitation: looks nothing like a conservation law
  Physics involves forces which set up stress which transmits the forces:
  How is this stress connected to the geometry ?

Variational Principes II: Auxiliary Variables

- H depends on $\mathbf{X}$ only through g_{ab} and K_{ab}

  Response of H to defo: $\mathbf{X} \to \mathbf{X} + \delta\mathbf{X}$ is through g_{ab} and K_{ab}

  Snag: they are not independent!
  Depend on $\mathbf{X}$ through the tangent vectors $\{\mathbf{e}_a, \mathbf{n}\}$

- Structural constraints connect them to $\mathbf{X}$:

$$g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b \,, \quad K_{ab} = \mathbf{e}_a \cdot \partial_b \mathbf{n}$$

$$\mathbf{e}_a = \partial_a \mathbf{X} \,, \quad \mathbf{e}_a \cdot \mathbf{n} = 0 \,, \quad \mathbf{n}^2 = 1$$

  Enforce using Lagrange multipliers:

$$\begin{aligned} H_C &= H[g_{ab}, K_{ab}] + \int dA \; \mathbf{f}^a \cdot (\mathbf{e}_a - \partial_a \mathbf{X}) \\ &\quad + \int dA \Big[\lambda_\perp^a (\mathbf{e}_a \cdot \mathbf{n}) + \lambda_n (\mathbf{n}^2 - 1) \Big] \\ &\quad + \int dA \Big[\Lambda^{ab} (K_{ab} - \mathbf{e}_a \cdot \partial_b \mathbf{n}) + \lambda^{ab} (g_{ab} - \mathbf{e}_a \cdot \mathbf{e}_b) \Big] \end{aligned}$$

Conservation laws

$$
\begin{aligned}
H_C \;=\; & H[g_{ab}, K_{ab}] + \int dA \; \mathbf{f}^a \cdot (\mathbf{e}_a - \partial_a \mathbf{X}) \\
& + \int dA \Big[\lambda_\perp^a (\mathbf{e}_a \cdot \mathbf{n}) + \lambda_n (\mathbf{n}^2 - 1) \Big] \\
& + \int dA \Big[\Lambda^{ab} (K_{ab} - \mathbf{e}_a \cdot \partial_b \mathbf{n}) + \lambda^{ab} (g_{ab} - \mathbf{e}_a \cdot \mathbf{e}_b) \Big]
\end{aligned}
$$

- Treat g_{ab}, K_{ab}, $\mathbf{e}_a$, $\mathbf{n}$ and $\mathbf{X}$ as independent variables

- Noether's theorem: Translational invariance of $H[\mathbf{X}]$ $\Rightarrow$ Existence of conserved stress tensor

- $\mathbf{X}$ only appears in the tangential constraint

  $H_C =$ [terms independent of $\mathbf{X}$] $+ \int dA \; \mathbf{f}^a \cdot (\mathbf{e}_a - \partial_a \mathbf{X})$

  $\Rightarrow \delta_{\mathbf{X}} H_C = \nabla_a \mathbf{f}^a$

- In Equilibrium $\mathbf{f}^a$ is conserved: $\nabla_a \mathbf{f}^a = 0$

  In general, $\mathbf{f}^a \neq 0$ even if $\nabla_a \mathbf{f}^a = 0$

- $\mathbf{f}^a$ will be identified as the stress tensor
  Transmits forces along the membrane, a 3×2 matrix

  $\mathbf{f}^1$ = force transmitted across $u^1 =$ constant

Stress depends only on the geometry

- Construction: Vary $\mathbf{e}_a$, $\mathbf{n}$, g_{ab} and K_{ab}

$$\mathbf{f}^a = (T^{ab} - H^{ac} K_c{}^b)\, \mathbf{e}_b - \nabla_b H^{ab}\, \mathbf{n}$$

(1) $T^{ab} \approx -2\delta H/\delta g_{ab}/\sqrt{g}$ not conserved
(2) $H^{ab} = \partial \mathcal{H}/\partial K_{ab}$

Remember: Energy density $\mathcal{H}$ is a functional of g_{ab} and K_{ab}

- Important point:
  Stress completely determined by membrane geometry

Stress in a fluid membrane:

- $\mathcal{H} = \kappa\, K^2/2 + \bar\kappa\, \det K^a{}_b + \sigma$, multiplier σ fixes $A \Rightarrow$

  $\mathbf{f}^a = f^{ab}\,\mathbf{e}_b + f^a\,\mathbf{n}$ where

$$f^{ab} = \kappa\, K(K^{ab} - \frac{K}{2} g^{ab}) - \sigma g^{ab}\,, \qquad f^a = -\kappa\,\nabla^a K$$

- f^{ab} local quadratic in K_{ab}; non-isotropic (not a perfect fluid)
  Lines of tangential stress $\parallel$ lines of curvature $\Rightarrow$
  Stress highs and lows correlate with curvatures

  Gaussian energy does not contribute explicitly: but $H^{ab} \neq 0$
  – as a result it does contribute to boundary conditions (see below)

- Normal stress requires curvature gradients

- $K = 0$ in minimal surfaces. No stress if tension is not applied

Euler-Lagrange from conservation law

- Divergence structure also mutilated in Euler-Lagrange equation

- $\nabla_a \mathbf{f}^a = P\,\mathbf{n}$ vs. $-2\nabla^2 K + \text{cubic in}\, K_{ab} = P$

  Discrepancy: three conservation laws vs. one shape equation?

- Project $\nabla_a \mathbf{f}^a = P\,\mathbf{n}$ $\perp$ and $\|$

  $\mathbf{f}^a = f^{ab}\mathbf{e}_b + f^a \mathbf{n}$

$$\begin{aligned} \nabla_a f^a - K_{ab} f^{ab} &= P \\ \nabla_a f^{ab} + K^b{}_a f^a &= 0 \end{aligned}$$

  Dismantle divergence:
  ($\perp$) is shape equation
  ($\|$) are Bianchi ids (consistency conditions)

Monge

- Simple structure mutilated in Monge gauge

- Height function, $\mathbf{r} \to h(\mathbf{r})$, small gradient $(\nabla h)^2 \ll 1$
  Adapted basis $\mathbf{E}_1, \mathbf{E}_2, \mathbf{k}$

  $$\mathbf{f}^i \sim (\kappa\, T_B^{ij} + \sigma\, T_M^{ij})\, \mathbf{E}_j + N^i \mathbf{k}$$

  $$T_B^{ij} = \nabla^2 h\, \nabla^i \nabla^j h - \frac{1}{2}(\nabla^2 h)^2 \delta^{ij} - \nabla^i (\nabla^2 h)\, \nabla^j h$$
  $$T_M^{ij} = \nabla^i h\, \nabla^j h - \frac{1}{2}(\nabla h)^2 \delta^{ij}$$

  $$N^i = \nabla^i \nabla^2 h$$

- $\nabla_i T^{ij} = 0$ when $\kappa(\nabla^2)^2 h - \sigma \nabla^2 h = 0$
  $T_B^{ij} \neq T_B^{ji}$ a gauge artifact; $\nabla_i T_{ji} \neq 0$

Boundary terms:

$$\begin{aligned} H_C &= H[g_{ab}, K_{ab}] + \int dA\ \mathbf{f}^a \cdot (\mathbf{e}_a - \partial_a \mathbf{X}) \\ &\quad + \int dA \Big[\lambda_\perp^a (\mathbf{e}_a \cdot \mathbf{n}) + \lambda_n (\mathbf{n}^2 - 1) \Big] \\ &\quad\quad + \int dA \Big[\Lambda^{ab} (K_{ab} - \mathbf{e}_a \cdot \partial_b \mathbf{n}) + \lambda^{ab} (g_{ab} - \mathbf{e}_a \cdot \mathbf{e}_b) \Big] \end{aligned}$$

The variations wrt both $\mathbf{X}$ and $\mathbf{n}$ come with boundary terms:

$\mathbf{X} \to \mathbf{X} + \delta \mathbf{X}$:

$$\delta H_C = \int dA\, \nabla_a \mathbf{f}^a \cdot \delta \mathbf{X} - \nabla_a \left[\mathbf{f}^a \cdot \delta \mathbf{X} \right]$$

$\mathbf{n} \to \mathbf{n} + \delta \mathbf{n}$ modulo $H^{ab} = -\Lambda^{ab}$:

$$\delta H_C = \mathsf{bla} + \int dA\, \nabla_a \left[H^{ab} \mathbf{e}_b \cdot \delta \mathbf{n} \right]$$

Contribution from K_G to $\mathbf{l} \cdot \delta \mathbf{n}$!

Interpretation of $\mathbf{f}^a$:

Let bulk be in equilibrium outside of N boundaries/embedded particles
Construct a curve around one of them, Γ say

- Under deformation, non-vanishing in neighborhood of Γ:

$$\delta H_C = -\oint_\Gamma ds\, l_a \mathbf{f}^a \cdot \delta \mathbf{X} + \oint_\Gamma ds\, l_a H^{ab} \mathbf{e}_b \cdot \delta \mathbf{n}$$

- Translate Γ: $\delta \mathbf{X} = \delta \mathbf{a}$, $\delta H_C = -\mathbf{a} \cdot \oint_\Gamma ds\, l_a \mathbf{f}^a$
  $l_a \mathbf{f}^a$ as the force/unit length acting on Γ

- Rotate Γ: $\delta \mathbf{X} = \mathbf{b} \times \mathbf{X}$, $\delta \mathbf{n} = \delta \mathbf{b} \times \mathbf{n}$,

$$\delta H_C = -\mathbf{b} \cdot \oint_\Gamma ds\, l_a \left[\mathbf{X} \times \mathbf{f}^a + H^{ab} \mathbf{e}_b \times \mathbf{n} \right] .$$

  Conserved Torque Tensor $\mathbf{m}^a = \mathbf{X} \times \mathbf{f}^a + H^{ab} \mathbf{e}_b \times \mathbf{n}$

Conformal stretching

- fix metric using local Lagrange multipliers $\mathcal{T}^{ab}$:

$$H_C \to H_C - \frac{1}{2}\int dA\, \mathcal{T}^{ab}(g_{ab} - \Omega(u)\, g_{ab}^{(0)})$$

  $f^{ab} \to f^{ab} + \mathcal{T}^{ab}$, $f^a \to f^a$
  Stress no longer completely determined by geometry!

- Euler-Lagrange:
  $(\perp)$ $\mathcal{E} = 0 \to \mathcal{E} - K_{ab}\mathcal{T}^{ab} = 0$
  $(\|)$ $\nabla_a \mathcal{T}^{ab} = 0$

  Stress associated with constraint acts as a source
  It is separately conserved

- Unstretchable surface: $\Omega = 1$ Paper folding!

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